Trade-off Method to Assess the Interaction Between Light Shelves and Complex Ceiling Forms for Optimized Daylighting Performance

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Recent research has shown that ceiling geometry can greatly affect the illuminance levels and daylighting uniformity in indoor spaces. Usually, modeling and evaluating the daylighting performance of flat ceilings and light shelves can be done easily. However, with the increase of sophisticated and complex parametric forms of suspended ceilings, architects have been struggling to adequately assess their daylight performance. This is mainly due to the time and effort required to examine all parameters within the ceiling form. Additionally, most available architectural tools for examining parametric ceilings are based on the use of Genetic Algorithm (GA), which proves to be an efficient way to search for an optimal solution but lacks the ability of providing the designer with multiple sets of alternatives for better decision making. In response, this paper examines a workflow based on the use of the Pareto principle that has been developed in order to assess the ceiling parameters and variables. It then ranks them according to their performance in search for optimized configurations that allow architects and designers a trade-off in the early design stage. Then, the paper will introduce a parameters elimination method to decrease the time required for daylighting simulations when new parameters are added to a ceiling that has been analyzed.

Keywords: Pareto principle, Genetic Algorithm (GA), performance based design, daylighting.

Introduction

For the past few decades, computational tools and techniques have been a fundamental part in architectural design. They have enabled architects to deal with forms that previously could not have been drawn or built before and that require non-standard engineering methods for their fabrication. Lately, there has been elevated attention toward parametric modeling. This is mainly due to the availability of the recently emerged visual parametric modeling tools, which hide the algorithmic complexity of a model behind an easy-to-use visual programming interface. Therefore, these decrease the technical skills required for the use of computation and contribute to its adoption in architectural design. Parametric tools facilitate the exploration of alternative designs within a single model using parameters and math formulas to control geometric and constructive aspects of the architectural model geometry (Terzidis 2012). With the rise of parametric tools, new complex architectural ceiling forms have emerged, because it was proved in a previous research study that ceiling forms can greatly affect the daylighting conditions in interior spaces (Freewan, Shao, and Riffat 2008), therefore, the exploration of using algorithmic and parametric design approaches to assess the daylighting performance of complex ceiling forms, as well as their interaction with light shelves, is necessary.

This paper reviews the existing approach of using Genetic Algorithm (GA) in optimizing architectural models with regard to their daylighting performance. Later, the paper will compare GA to the workflow set forth in this study, which is based on the use of visual Pareto charts for better analysis of the ceiling's parameters and their effect on the ceiling's daylighting performance.

Methodology

This paper aims at establishing a framework to assess the performance of non-uniform ceiling geometry through parameters ranking. The study first sets up a simple model within a parametric environment, creating the initial ceiling geometry with various sets of parameters. Changing those parameters results in the creation of multiple ceiling forms. Daylighting performance of each ceiling configuration is evaluated using a brute-force algorithm. As a straightforward approach to problem solving, a brute-force solver will generate all possible parameters combinations (Rosen 2011) and solve the definition for each combination, allowing the exploration of all configurations within a parametric model. A DIVA component for Grasshopper is used for calculating illuminance values for each ceiling configuration. Results are automatically exported to an Excel spreadsheet. The results are then analyzed and sorted according to their daylighting performance based on a pre-set fitness function value. A Pareto chart is created to determine which parameters contribute to improving the daylighting performance of the ceiling. A Pareto chart, also called a Pareto distribution diagram, is a vertical bar graph in which parameters are plotted in decreasing order of relative frequency from left to right. Pareto charts are extremely useful for analyzing which parameter has induced a specific pre-set objective function, which in this case is a calculation of useful daylight illuminance values. The taller bars on the chart, which represent frequency, clearly illustrate which variables have the greatest cumulative effect on daylighting performance.

The Pareto chart provides a graphic visualization of the Pareto principle, also known as the 80-20 rule, which is a theory maintaining that 80% of the output in a given situation or system is produced by 20% of the input (Ramachandran and Tsokos 2009). Using a Pareto chart allows designers to find the optimum configuration through a trade-off method rather than searching for a single optimized solution (Coello, Van Veldhuizen, and Lamont 2002). Figure 1 shows the workflow used for the purpose of this study.

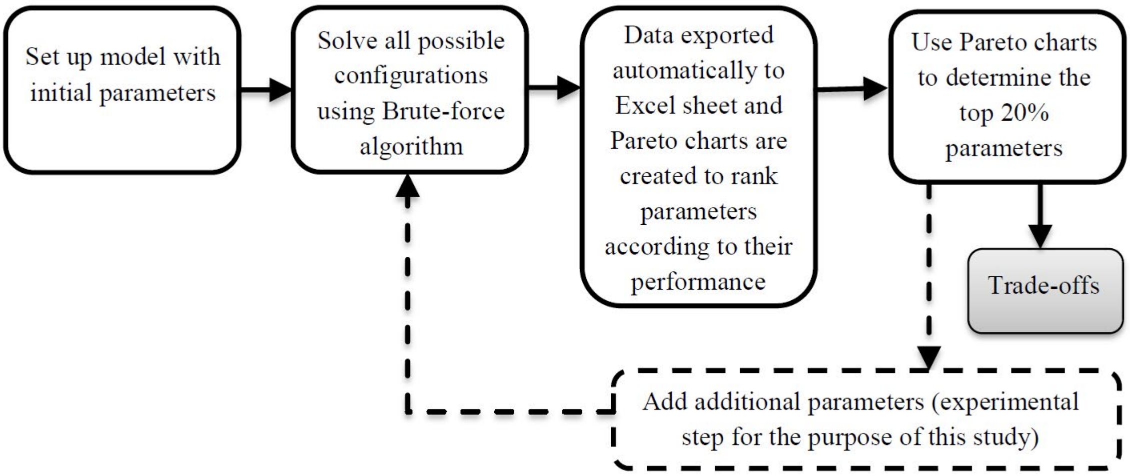

Figure 1. Workflow of the framework used to determine optimized ceiling configurations. Dashed outline shows extra steps used to examine the Pareto ranking of the chosen top 20% when adding new parameters to the original ceiling geometry.

Model setup

For the purpose of this study, a simplified test room is modeled in a parametric environment, Grasshopper for Rhino. The depth and width of the test room are 9m and 6m, respectively. Light shelves are useful primarily with windows that have large glazing areas and heights greater than 2.2 meters (Wulfinghoff 1999), so it was decided to use a 2.5m high south-oriented window for maximized light shelf performance. The location is fixed to Miami, Florida, USA (25.7877° N, 80.2241°). This location has annual sunlight availability of 66% (Corporation et al. 1978). A parametric light shelf is modeled and placed at 2m high. Controlling the light shelf parametrically allowed for alternations between a flat and a curved light shelf. Since light falling on a light shelf is redirected toward the ceiling and then reflected again into the interior space, a high ceiling reflectance (R=0.85) is used to maximize light reflection. Interior wall and floor reflectance value is set to 55%. An analysis grid that contains 140 nodes across the room is placed at 2.5m.

Daylighting performance objective function

A fitness function is used to test each possible configuration. During initial daylighting simulations illuminance levels (lux), and Daylight Autonomy (DA), a climate-based daylighting metric, were both used to develop two fitness functions. The Radiance ambient bounce parameter was set to a value of 7. The initial DA simulation took 17 minutes to complete, in contrast to the illuminance level simulation which took 3 minutes to complete, this is mainly because the calculation procedure of DA determines the hourly illuminance level in all 140 analysis grid points in the test room for the entire year (Reinhart, Mardaljevic, and Rogers 2006). Running simulations for a large number of iterations can be a time consuming process. To overcome this problem, illuminance level simulations are used. The time and date is set to June 21 at 12:00 , a time of day when shading is strongly desired. The Illuminating Engineering Society of North America (IES) recommends illuminance values ranging from 300 to 1500 lux in various types of spaces (DiLaura and America 2011). Accordingly, the fitness function is set to be the percentage of the room area with illuminance levels between 300 and 1500 lux. DA values are adopted later for testing best and worst case scenarios and for the framework validation.

Ceiling geometry

The initial modeled ceiling geometry is a curved ceiling that has a third degree NURB curve profile. The profile curve is parametrically altered by changing the curve control points or their respective weights (Farin 1999). Increasing the value of the weight pulls the curve toward the respective control point while decreasing the value of the weight pushes the curve away from the respective control point. The profile curve of the ceiling is parametrically controlled by changing the curve weights set. A third degree curve usually has four sets of control points and four sets of weight. For each curve weights set three values are used (1, 2, and 3), resulting in 82 possible ceiling configurations. Figure 2a shows the initial ceiling modeled and an example of the profile curves that resulted from changing curve weights. Each possible ceiling form is paired with a flat light shelf and a curved one, thus a total of 164 configurations can be produced.

This study is targeted toward examining the performance of ceiling geometries in early design stages when architects test and compare conceptual designs by iteratively altering design parameters. However, architects often redefine their conceptual ideas for many reasons, such as aesthetic aspects, optimizing building performance, or client requirements. In a real world scenario, after testing the daylighting performance of all 164 configurations within the initial ceiling form, a designer might decide to add additional parameters to the ceiling form which, in turn, produce even more configurations. This would make the process of investigating the performance of the new parametric ceiling form a more time consuming process for the professional designer, which is problematic when a conceptual design phase is time constrained, as often is the case. In an effort to reduce the time required to run a daylighting simulation for the redefined surface, a possible method to apply to such a scenario is developed and validated. The method suggests that when additional parameters are introduced to an already examined ceiling form, some in-place parameters can be eliminated following examination of the produced visual Pareto chart, and only the parameters that drive performance should be used. This can substantially reduce the amount of the configuration produced by adding a new set of parameters to a parametric ceiling geometry, therefore reducing the time required for daylighting simulations.

For the suggested method validation, a new set of parameters is introduced to transform the initial ceiling form. The set of parameters are the rotation angles of two curves that are added to the lofted ceiling surface (Figure 2b). Each curve is rotated around the x-coordinate in two different angles. One curve is rotated 100˚ or 150˚; the second curve is rotated 150˚ or 225˚. Adding the new NURB curves with different rotation angles means that there will be 656 possible configurations within this redefined geometry of the phase II ceiling. For a more complex ceiling geometry, two more variables are added (Figure 2c). The surface is divided into 9 or 18 equal parts in the U and V directions, then transformed into triangulated panels. This makes the final possible variants within this geometry of the phase III ceiling go up to 1312. The surfaces of phases I, II, and III were all tested in terms of their daylighting performance using the workflow in Figure 1. Then a method of parameters elimination is introduced and validated.

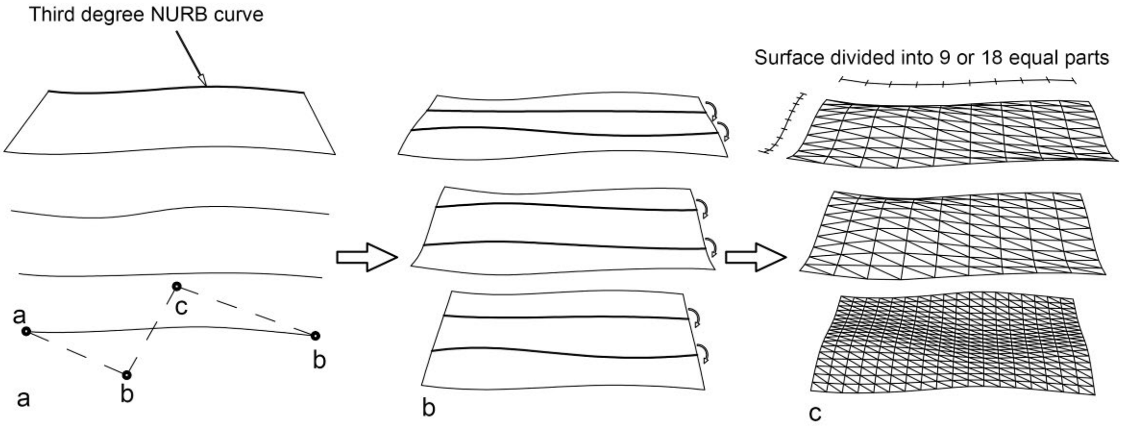

Figure 2. a) Phase I: ceiling surface and representation of multiple profile curves—a, b, c, and d represent curve control points. b) Phase II: two duplicate curves added and rotated around the X coordinate. c) Phase III: ceiling divided in the V-direction and the U-direction.

Genetic Algorithms versus Pareto principle

Genetic algorithms (GAs) are able to find single optimal solutions by using many of the principles of evolution that can be observed in nature. These include selection, crossover, and mutation. For the last few years, GA has been used as a tool for multidisciplinary design optimization (MDO). Extensive research studies have employed GA to optimize building performance. A research study conducted by Caldas and Norford applied the use of GA to manipulate window size and placement using energy simulations for optimized energy performance in buildings (Caldas and Norford 2002). A similar research study, conducted by Yi and Malkawi, explored building forms in relation to their energy performance (Yi and Malkawi 2009). Furthermore, GA was recently used for a parametric façades study. Recent research utilized GA to study façade optimization in the areas of daylight availability, glare reduction, and energy consumptions (Brotas and Rusovan 2012).

Although GA has proved to be an efficient way to search for an optimal solution, it lacks the ability to provide the designer with multiple sets of alternatives for a better decision-making process in the early stages of design. A drawback of using GA within Grasshopper is that the evolutionary solver available for Grasshopper searches by default for a solution that is "better" only in comparison to other presently known solutions; for this reason, evolutionary algorithms are best utilized on problems where it is difficult or impossible to test for optimality. This also means that an evolutionary algorithm never knows for certain when to stop, or how many iterations one wishes to explore. The study has examined the use of GA to search for an optimal ceiling configuration in regards to the best daylighting performance by using the built in evolutionary solver component within Grasshopper. The ceiling test had 656 possible configurations and the solver was able to find the optimal solution after the 8th generation, with 50 iterations in each generation. That means that the solver was able to find the best possible ceiling configuration after calculating the daylighting performance of 400 different variants of the ceiling geometry. However, the solver did not know when to stop; the author forced the solver to stop at the 40th generation (Figure 3). Another drawback of using the solver within Grasshopper to run daylighting simulation is that, by default, Radiance runs with stochastic randomness turned on, so the rays that are sent out during each simulation can vary from run to run, causing the results to vary greatly each time a configuration is tested, this effect is amplified the lower the Radiance's settings are. Therefore, the output results, which can be exported to Excel, will have duplicated configurations with different fitness values. This can be confusing to professional designers.

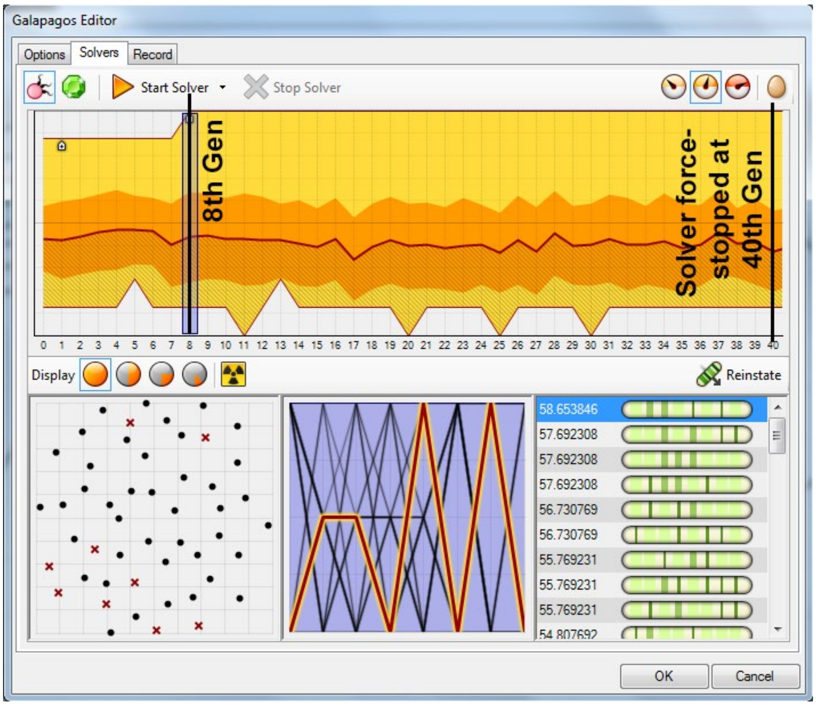

Figure 3. The evolutionary solver in Grasshopper was able to find an optimal solution after the 8th generation but was not able to stop on its own.

For the past decade, various research papers have focused on the use of the Pareto principle in multiple fields. An example of using Pareto principle in engineering can be seen in a research study conducted in order to optimize the design of a water distribution system. Rather than searching for a single solution with the best scalar fitness value, a Pareto principle based chart was used to plot a set of the collected data to allow for diverse solutions, which together defined the best possible multi-objective trade-off (Farmani, Savic, and Walters 2005). A recent research study introduced a visual approach based on the Pareto principle that ranks a large set of variables in search for an optimized set of configurations to maximize building energy efficiency and allow for trade-offs between design objectives and variables (Brownlee and Wright 2012).

Since Genetic Algorithms search for a single optimal solution, consequently an architect will not be able to perform trade-offs for alternative design solutions. Previous research has suggested that trade-off workflow is a fundamental keystone in providing suitable feedback for the decision making process (Radford and Gero 1980). A better workflow for testing and analyzing multiple configurations would be the Pareto principle, which ranks parameters visually according to their performance, therefore facilitating trade-offs.

Daylighting simulations

Daylighting simulation for all three phases was performed using the DIVA component for Grasshopper. All variables were connected to a brute-force Grasshopper component to calculate the percentage of the room area with illuminance values between 300 and 1500 lux. A brute-force algorithm computes every combination of the variables connected to the component in a simple, iterative manner.

An Excel component was configured to export each configuration's variables and their objective values—in this case, the percentage of the room area with useful illuminance values—to a spreadsheet for data analysis and the visual ranking of parameters using a Pareto chart. The interaction between the light shelf and the ceiling was examined in all three phases. As shown in Figure 4, a flat light shelf proved to have a better daylighting performance than a curved one. For example, a total of 82 configurations in phase I, with a flat light shelf, contributed 67% to 77% useful illuminance values to the room, with the median value of 71%. However, when a the flat light shelf was replaced with a curved one the percentage of the room area with useful illuminance for all 84 configurations varied between 65% and 75%, with the median value of 68.5%. In phase II, the median value for all 656 configurations when using a flat shelf and a curved one are 72% and 69.3%, respectively. Lastly, in phase III, the daylighting performance median value was increased by 6.5% when a flat light shelf was used.

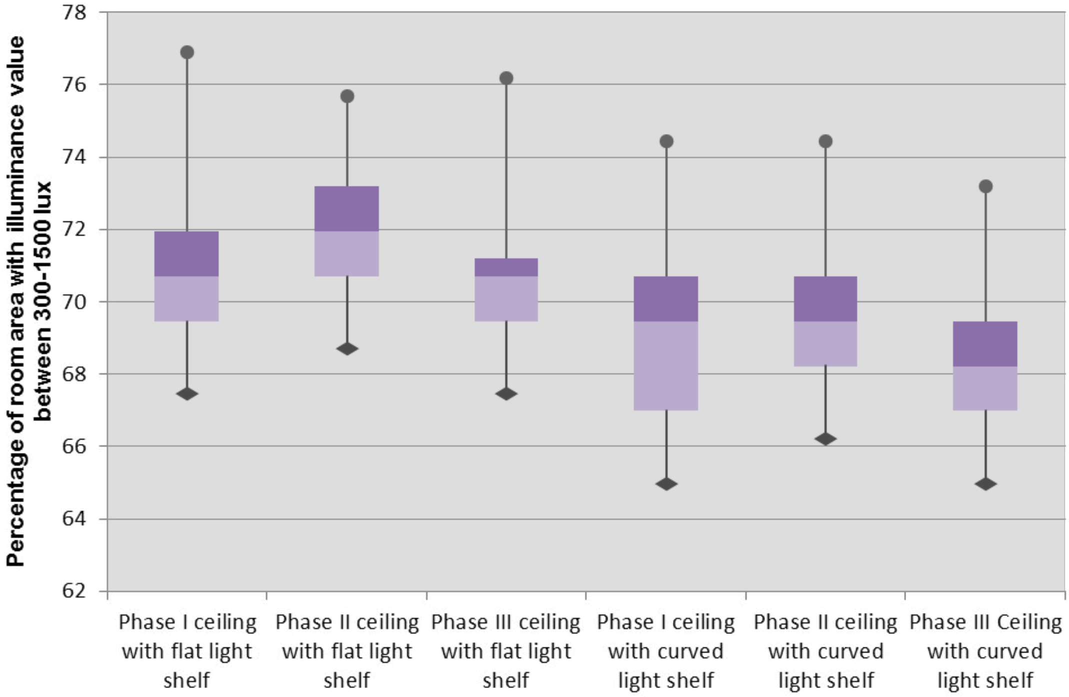

Figure 4. The performance of the flat and curved light shelf plotted for all combinations in each phase. Minimum values are represented with a diamond, and maximum values with a circle. The lines between the dark and light colored areas represent median values. The top and bottom of each box represent the 75th and 25th percentile, respectively.

Since daylight autonomy (DA) is an annual calculation, which means the entire year is taken into account, it has a great advantage in assessing the quantitative daylighting performance of a configuration in contrast to measuring illuminance value at a specific time and day. Therefore, the DA of the best scenario in each phase is calculated for using both a flat and a curved light shelf. The DA value 10m away from the window when using a flat and a curved light shelf are 44 and 40 respectively; at 8m from the window, the DA value is 60 and 42 respectively (Figure 5).

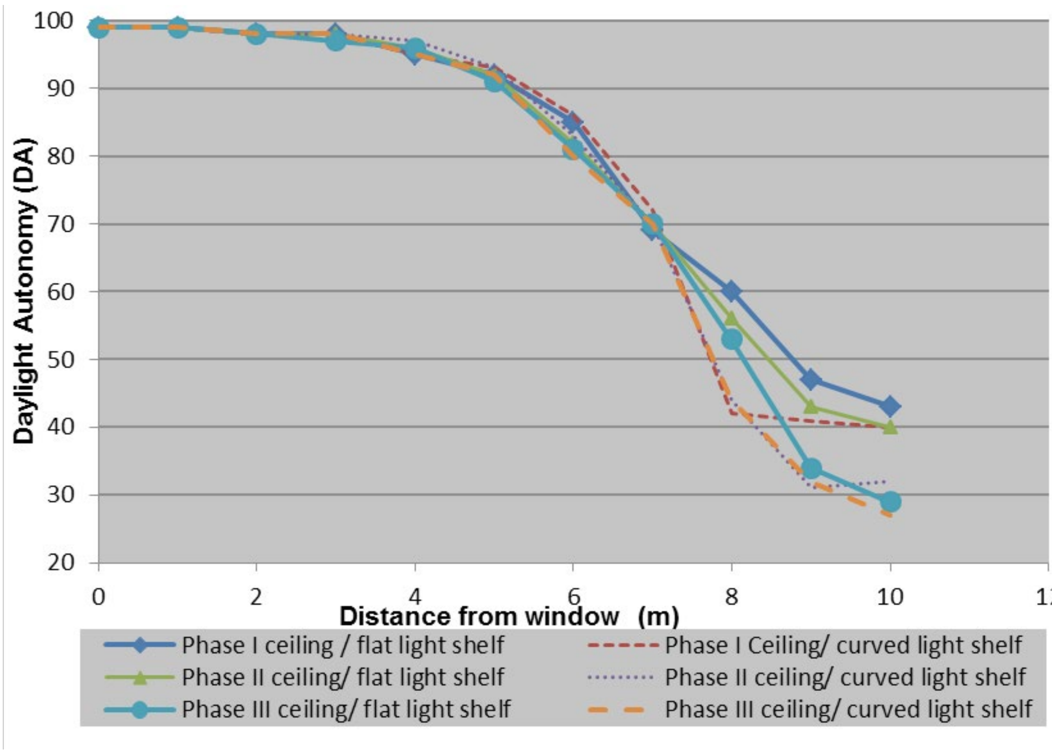

Figure 5. DA values at different points across the window for the best scenario in each phase

Ranking parameters using Pareto principle

Parameters from each phase are assessed visually using a Pareto chart. Parameters from phases I, II, and III include a profile curve with four different weight values that vary between 1 and 3, producing 81 different curve profiles (Table 1). To better sort all 81 possible curve profiles, the curve weight variance was calculated according to the following equation:

$$s^2 = \frac{\Sigma\left[\left(xi - \bar{x}\right)2\right]}{n - 1} s^2 = \frac{\Sigma\left[\left(xi - \bar{x}\right)2\right]}{n - 1}$$

Where

- s^2 = variance
- xi = every term in the weights set
- $\bar{x}$ = mean, the average of all the numbers in the weights set
- n = number of terms in the weights set

Solving the above equation helped in dividing the large group of 81 profiles into 7 subgroups for an easy-to-read Pareto chart. The subgroups are curves with variance values of 0, 0.25, 0.33, 0.67, 0.92, 1, and 1.3. The equation was integrated into the custom Excel spreadsheet to automate the Pareto chart creation workflow.

Phase	Parameters	Number of possible configurations
Phase I	Flat or curved light shelf Four curve weights, each weigh has a value of 1, 2, or 3	164
Phase II	In addition to parameters from phase I: A curve added with rotation angle 150˚ or 100˚ A second curve added with rotation angle 150˚ or 225˚	656
Phase III	In addition to parameters from phases I and II: Surface divided into 9 or 18 parts in the U and V directions and then transformed into triangulated panels	1312

Table 1. Parameters and numbers of possible ceiling and light shelf configurations for each phase.

Parameters from phase II were sorted according to the rotation angles of the curves added to the initial ceiling form. Again, those sets were divided into two groups: curves rotated at the same directions and curves rotated at opposite directions.

Parameters that drive performance are easily determined visually by examining the Pareto charts created (Figures 6, 7 and 8). Using the 80-20 rule, a line is plotted on each chart at 80%. The set of the parameters to the left of line are considered the best parameters that drive high daylighting performance, and the set of the parameters to the right have insignificant effect in improving performance. Additionally, a taller graph indicates that the parameter contributes to increasing the ceiling's daylighting performance. Ultimately, a Pareto chart can be a great asset for architectural model optimization mainly because it allows architects and designers a chance to trade off variables for an optimized building design.

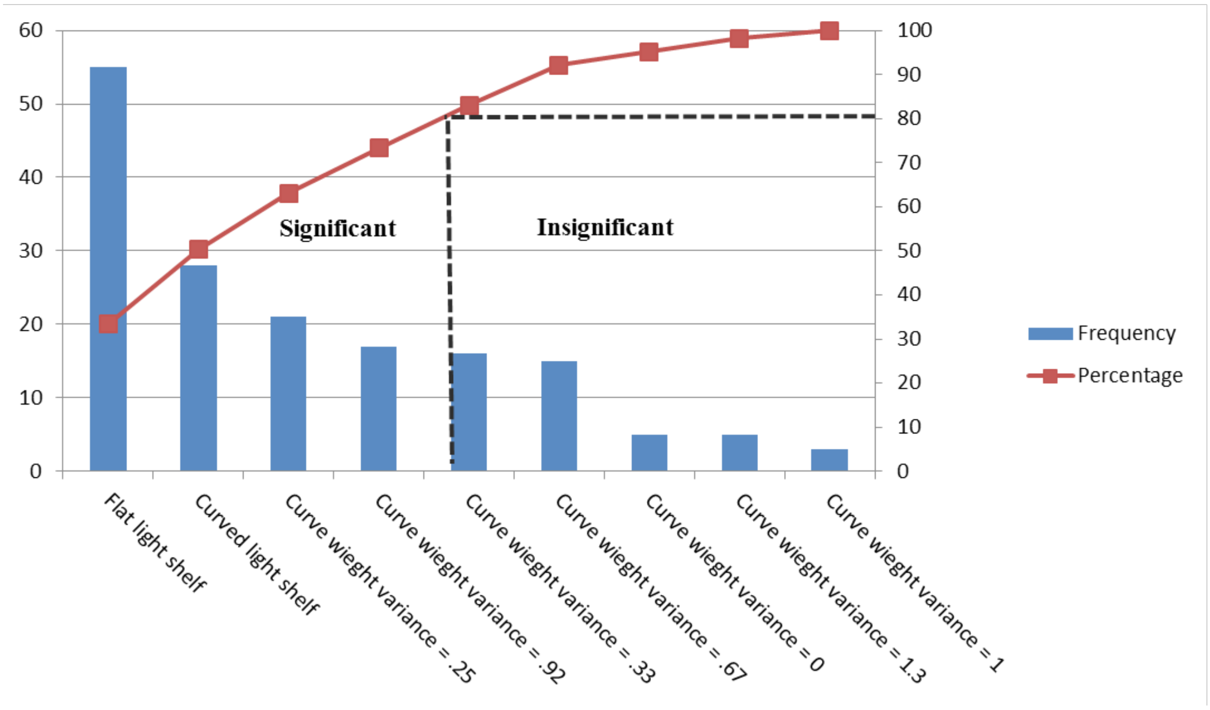

Figure 6. A Pareto chart ranking phase I parameters according to their performance. Parameters to the left of the dashed line drive performance; parameters to right of the dashed line have insignificant effect on performance.

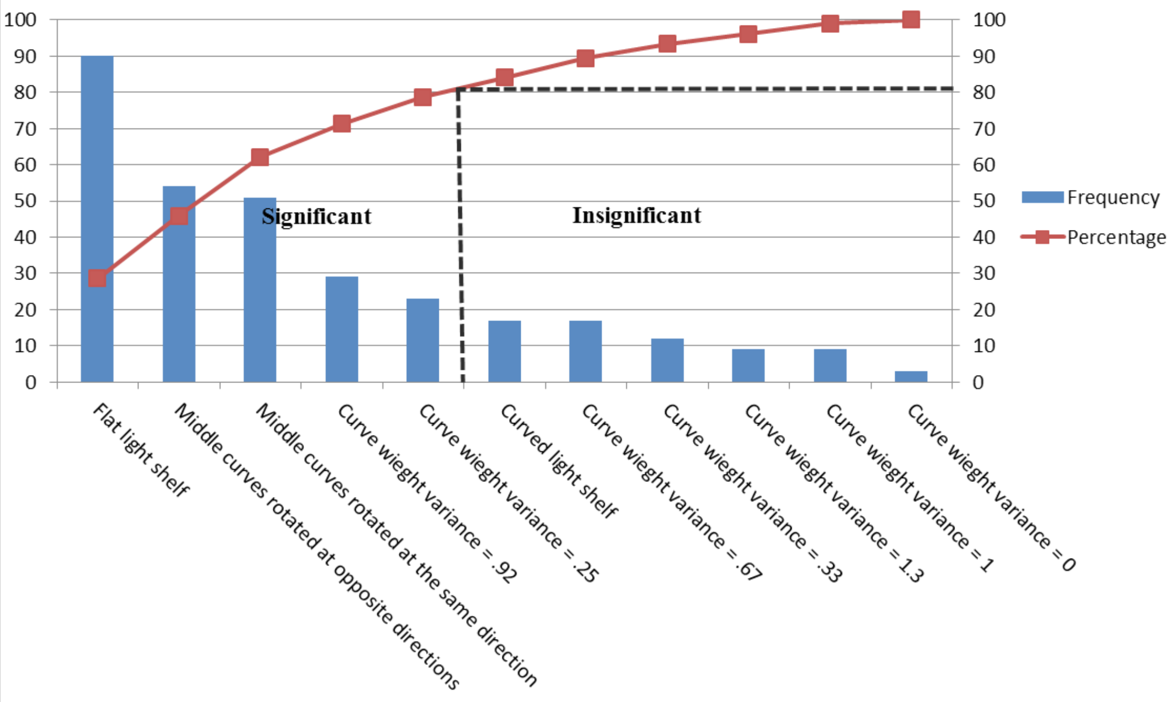

Figure 7. A Pareto chart ranking phase II parameters according to their performance. Parameters to the left of the dashed line drive performance; parameters to right of the dashed line have insignificant effect on performance.

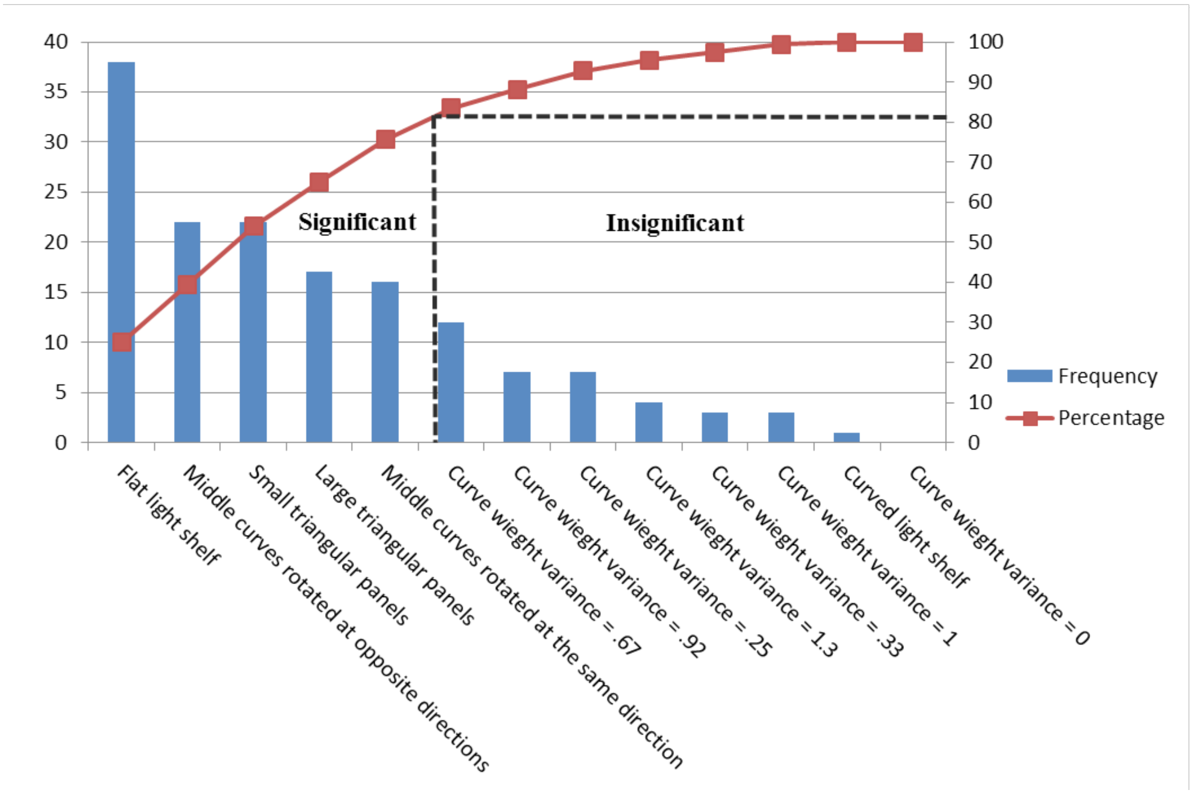

Figure 8. A Pareto chart ranking phase III parameters according to their performance. Parameters to the left of the dashed line drive performance; parameters to right of the dashed line have insignificant effect on performance.

Workflow validation

Upon examining the Pareto charts, the set of parameters driving daylighting performance are determined and evaluated in six different scenarios. Additionally, a method of parameters elimination is used when transforming the ceiling to a much more complex form.

Adding a set of parameters to the phase I ceiling caused an increase in the possible configurations from 164 to 656. Similarly, adding parameters to the phase II ceiling produced 1312 configurations. An architect starts by creating a conceptual design and then running a daylighting simulation to assess the initial ceiling design. The time involved in running such a simulation is not likely trivial. Sometimes architects expand their initial ideas by adding more parameters to their conceptual idea. Doing so greatly increases the time required for daylighting simulations, making it impossible to evaluate their new design option. This research study suggests that eliminating parameters that have the lowest bar height in the Pareto charts and using only the parameters that drive performance can sustainably reduce the time required to examine the architect's new design option. To validate this methodology, a set of daylighting simulations was conducted for six scenarios using the parameters elimination methodology.

After initially evaluating the daylighting performance for all configurations within phase I, parameters with taller bar graphs to the left of the 80-20 rule line were chosen to create a scenario and evaluate its performance with a flat and a curved light shelf. According to the Pareto chart in Figure 6, a ceiling profile curve with a weight variance value of 0.25 contributes to a higher ceiling performance. Therefore, a configuration using a ceiling profile curve with weights 1, 2, 1, and 1, a variance of 0.25 was evaluated. Daylighting simulation showed higher DA values across the back of the room when a flat light shelf was used, with a mean DA of 79% in contrast to a mean DA of 76% when a curved light shelf was used. A scenario with a parameter processing the lowest objective function was chosen to create a worst case scenario and evaluate its performance. A ceiling with a profile curve with weight variance value of 1, which according to the Pareto charts is the worst parameter in terms of objective performance, was paired with a flat light shelf and a curved one and daylighting simulation is performed. When a flat light shelf is used, the average DA across the test room is determined to be 73%, and a curved light shelf produced a lower average DA value of 70% (Figures 9a and 9b). It is clear that using parameters with taller bars and following the left of the 80-20 rule line produced a configuration with better daylighting performance compared to using parameters to the right of the 80-20 rule line.

In another scenario, the best parameters in the phase II ceiling were chosen to model a configuration with high daylighting performance. The parameters used were a profile curve with weights 1, 3, 1, and 2, where weight variance was equal to 0.92 and two middle curves rotated at opposite directions with angle values of 225° and 150°. This scenario was examined using flat and curved light shelves and the mean DA was calculated. Similar to the first two scenarios from phase I, the flat light shelf contributed to a better daylighting quantity across the room with an average DA of 82% while the curved light shelf decreased the ceiling performance with a lower DA average of 76% (Figure 9c). As shown in Figure 9d, parameters from the far right on the Pareto chart were used to create a worst case scenario; lower DA values were recorded.

Lastly, two more scenarios were modeled using "best" and "worst" parameters from the phase III Pareto chart. The parameters used for the first scenario were a profile curve that had a weight variance of 0.67, with two curves rotated at angles 225° and 150°, and small triangular panels dividing the surface into 18 parts in the U and V directions. The second scenario used parameters that drove the lowest objective function; a profile curve that had a weight variance of 1, with two curves rotated 150° and 150°, and large triangular panels. Using parameters from the far right of the Pareto chart contributed to a 10% reduction in the average DA when a flat light shelf was used, and 8% when a curved light shelf was used (Figures 9e and 9f).

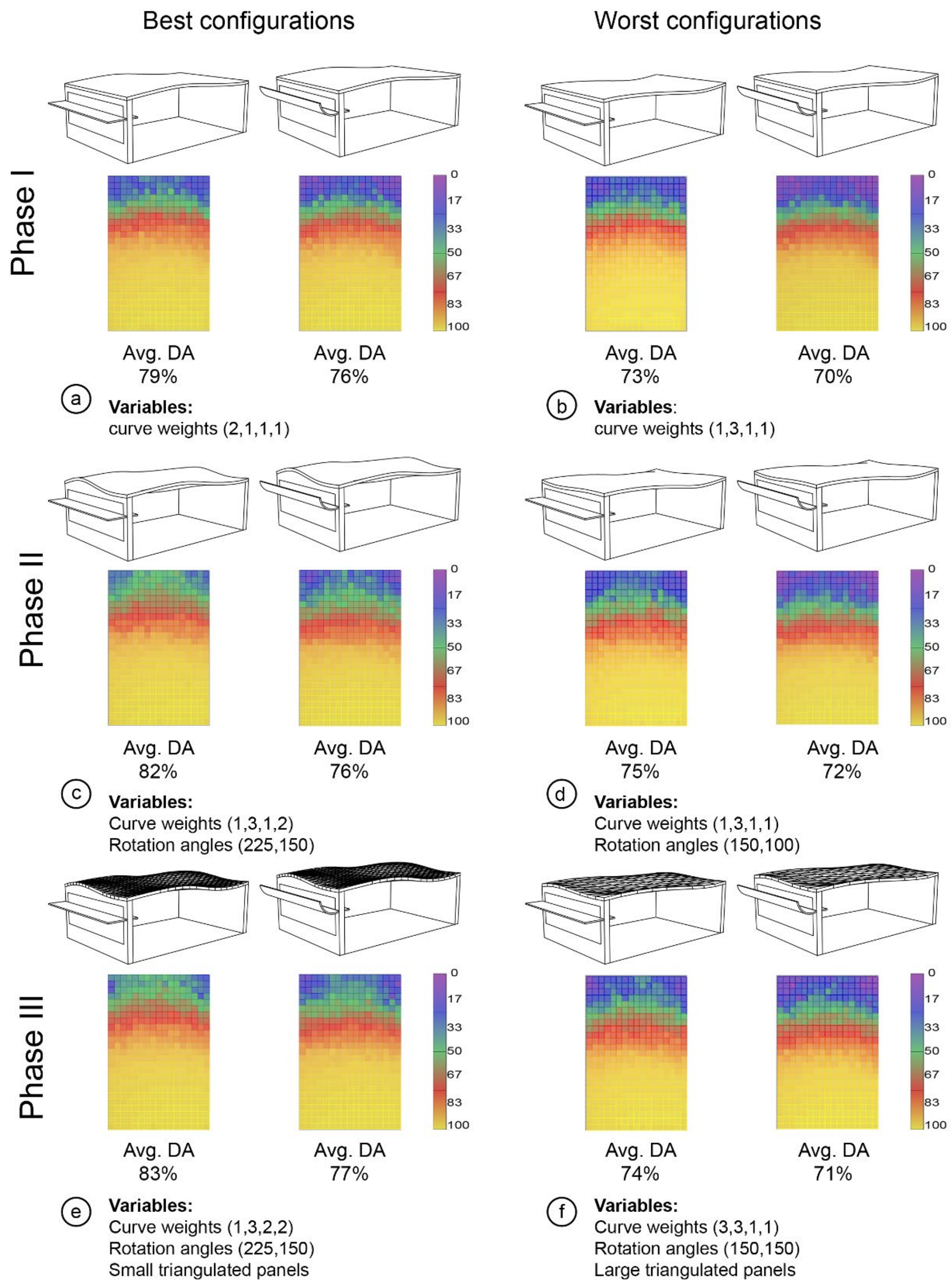

Figure 9. Phases I, II, and III: best and worst configurations daylighting analysis.

Parameters eliminations method

The method of parameters elimination can speed up the workflow of evaluating architectural model performance when a designer decides to redefine an already evaluated geometry. An example of the parameters elimination method is applied on the scenarios introduced in this research study. After carefully examining the Pareto chart for the phase I and phase II ceilings, it was determined that the top two weight variance parameters (weight variance of 0.25 and 0.92) remained in the same position in both charts, meaning that they remained the top parameters that contributed to a high objective function. During the process of generatively transforming the phase I ceiling into a phase II ceiling, these two top performance parameters can be used and the rest of parameters can be eliminated, thus reducing the possible configurations to be evaluated in the phase II ceiling from 656 to 88 configurations, which greatly reduces the time required to run daylighting simulations. Similarly, when evaluating the performance of the phase III ceiling, most of the top parameters in phase II remained in the same position, but only one parameter, the weight variance parameter with a value of 0.67, moved to the left of the two top weight variance parameters with values of 0.25 and 0.92. However, a deeper look revealed that all weight variance parameters moved to the right of the 80-20 rule line, meaning that they contributed to insignificant changes in the ceiling's performance. Again, if an architect decides to generatively transform a phase II ceiling into a triangulated panel ceiling, a great amount of time and effort usually required for daylighting simulations can be greatly reduced when the best parameters from the phase II chart are used and the rest of the parameters are eliminated, thus reducing the possible configuration from 1312 to 176 configurations.

Conclusion and discussion

When evaluating the performance of complex ceiling geometries with a large number of variables, using the Genetic Algorithm solver in Grasshopper can be difficult and confusing. Additionally, the outcome is only a single optimal configuration, making it impossible to use a trade-off process in early design stages. This research study introduced the use of visual Pareto charts to facilitate trade-offs and distinguish between parameters that drive performance and parameters that have insignificant effect on the ceiling's performance, which can be determined based on the location and the bar graph length of each parameter on the chart. Furthermore, the relationship between flat and curved light shelves and ceiling geometry was examined in terms of quantitative daylighting performance. It was determined that flat light shelves performed slightly better than curved light shelves. Designers usually redefine their initial designs by adding parameters to the existing geometry, a process also known as generative modeling. A process of parameter elimination was used to greatly reduce the amount of configurations to be tested, therefore reducing the time and effort required to examine the daylighting performance of redefined ceiling geometry. Further work is needed to extend this methodology for even more complex geometry with larger amounts of variables. The parameters elimination method utilized deemed only "viable" for the ceiling geometry examined in this research study.

While the workflow presented in this study examined the optimization of a single aspect of the ceiling geometry, daylighting performance, early conceptual design requires assessment of more than one objective function. Therefore, the utilized workflow should be further explored to integrate the possibility of using the Pareto principle along with parameters elimination process to assess architectural ceilings for multi-objective optimization.

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